Expected Shapley-Like Scores of Boolean Functions: Complexity and Applications to Probabilistic Databases

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Theoretical motivation

The tractability landscapes of Shapley value computation and probabilistic query evaluation are similar – How does the Shapley value computation landscape change when the database is probabilistic?

Shapley-like scores

- V: finite set of Boolean variables
- $\varphi : 2^V \to \{0, 1\}$ Boolean function over V
- $c : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$: coefficient function (assumed to have PTIME evaluation when input in unary)

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$$\mathrm{Score}_c(\varphi, V, x) \stackrel{\text{def}}{=} \sum_{E \subseteq V \setminus \{x\}} c(|V|, |E|) \times \left[\varphi(E \cup \{x\}) - \varphi(E)\right].$$

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Example

- $c_{\text{Shapley}}(k, \ell) \stackrel{\text{def}}{=} \frac{\ell!(k-l-1)!}{k!} = \binom{k-1}{l}^{-1} k^{-1}$: Shapley value [Shapley et al., 1953]
- $c_{\text{Banzhaf}}(k, \ell) \stackrel{\text{def}}{=} 1$: Banzhaf value [Banzhaf III, 1964]
- $c_{\text{PB}}(k, \ell) \stackrel{\text{def}}{=} 2^{-k+1}$: Penrose–Banzhaf power [Kirsch and Langner, 2010]

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- $\text{EScore}_c(\varphi, x) \stackrel{\text{def}}{=} \sum_{\substack{Z \subseteq V \\ x \in Z}} (\Pr(Z) \times \text{Score}_c(\varphi, Z, x))$ the expected score of x for φ

Problems studied

We consider classes of representations of Boolean functions, e.g., Boolean circuits, d-D circuits. We assume $\varphi(\emptyset)$ to be computable in PTIME.

- $\mathsf{EV}(\mathcal{F}) : \varphi \in \mathcal{F} \mapsto \Pr(\varphi)$
- $\mathsf{Score}_c(\mathcal{F}) : (\varphi \in \mathcal{F}, x \in V) \mapsto \mathsf{Score}_c(\varphi, V, x)$ for some coefficient function c
- $\mathsf{EScore}_c(\mathcal{F}) : (\varphi \in \mathcal{F}, x \in V) \mapsto \mathsf{EScore}_c(\varphi, x)$

We look for the complexity of these problems and for (Turing) polynomial-time reductions between problems, denoted $A \leqslant_{\mathrm{P}} B$, for class of Boolean functions (and $A \equiv_{\mathrm{P}} B$ for two-way reductions).

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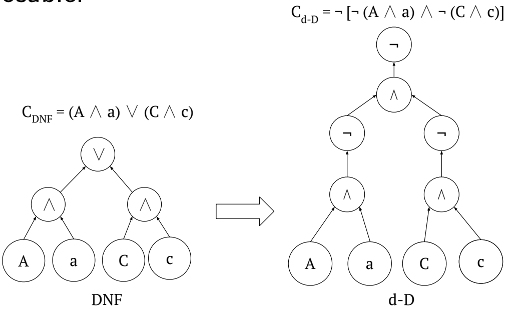

Expected Shapley-like scores

Area

ID	Region	Area	Prob.	Prov.
01	Valparaiso	16,000	0.4	*A*
02	Atacama	75,000	0.3	*B*
03	Metropolitan	15,000	0.6	*C*
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Density

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$x \in V$	p_x	$\text{Score}_{c_{\text{Shapley}}}(\varphi_{\text{ex}}, V, x)$	$\text{EScore}_{c_{\text{Shapley}}}(\varphi_{\text{ex}}, x)$
A	0.4	0.25	0.076
a	0.5	0.25	0.076
C	0.6	0.25	0.216
c	0.8	0.25	0.216
		1.0	0.584

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- $\mathrm{Score}_c(\mathcal{F}) \leqslant_{\mathrm{P}} \mathrm{EScore}_c(\mathcal{F})$ for any $\mathcal{F}$, c: just compute EScore_c with all probabilities set to 1
- $\mathrm{Score}_{c_{\mathrm{Shapley}}}(\mathcal{F}) \equiv_{\mathrm{P}} \mathrm{EV}(\mathcal{F})$ for any class $\mathcal{F}$ closed under $\vee$-substitutions [Kara et al., 2024] and when probabilities are uniform (unweighted model counting)

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Theorem

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The tractability landscape of $\mathrm{EScore}_{c_{\mathrm{Shapley}}}$ (and $\mathrm{EScore}_{c_{\mathrm{Banzhaf}}}$ under a mild condition) is exactly the same as that of EV

Exact algorithms

In the case where we have a d-D C, possible to design specific algorithms (extending those of [Deutch et al., 2022, Abramovich et al., 2023]) for EScore_c with complexity (ignoring arithmetic costs):

- $O\left(|C| \times |V|^5 + \mathrm{T}_c(|V|) \times |V|^2\right)$ where $\mathrm{T}_c(\alpha)$ is the cost of computing the coefficient function on inputs $\leqslant \alpha$

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- $O(|C| \times |V|)$ for c_{Banzhaf}

Application to probabilistic databases

- TID database, Boolean query q in some query language
- Define Score_c, EScore_c of a tuple for a query as Score_c, EScore_c of the Boolean provenance of the query over the database
- We compare to PQE (Probabilistic Query Evaluation, i.e., computing the probability of a Boolean query)

Theorem

- $\mathrm{EScore}_c(q) \leqslant_{\mathrm{P}} \mathrm{PQE}(q)$ *for any c, query q (whatever the query language!)*
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We inherit all tractability and intractability results for PQE, e.g., dichotomy for UCQs [Dalvi and Suciu, 2013] or queries closed under homomorphisms [Amarilli, 2023]

Set-up

- Implementation of all algorithms within ProvSQL
- Same experimental set-up as in [Deutch et al., 2022]: 1 GB TPC-H database, 8 TPC-H queries with some adaptations (e.g., removing aggregates), computation of Shapley/Banzhaf scores for all input tuples
- Non-Boolean queries: computation for every output tuple
- Proof-of-feasibility rather than in-depth experiments
- Compilation to d-D:
  - Check whether Boolean circuit is already an independent circuit
  - Otherwise, try to find a low-treewidth decomposition of the circuit, and use it to build a d-D
  - Otherwise, use an external knowledge compiler (but never required)

Results

# Output tuples	Provenance time (s)	Compilation time (s)	Shapley time (s) Determ.	Shapley time (s) Expect.	Banzhaf time (s)
11620	2.125	1.226	0.762	1.758	0.467
5	1.117	0.044	0.766	40.910	0.191
4	1.215	0.017	0.269	9.381	0.085
1783	1.229	0.018	0.023	0.037	0.015
61	0.174	0.001	0.001	0.002	0.001
466	0.247	0.084	0.159	0.455	0.094
91159	2.711	0.749	0.655	1.008	0.489
56	1.223	0.000	0.000	0.000	0.000

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Very encouraging! Shapley value computation does not have such a huge overhead!

Main message

- Expected Shapley value computation is not (much) more costly than probabilistic query evaluation
- Landscape seems clearer than for deterministic Shapley value computation
- PQE (and Expected Shapley value computation) is quite feasible in practice, even on large datasets
- Connection to SHAP-score [Van den Broeck et al., 2022] is not quite clear (there is also a probability distribution, but not used in the same way)
- What are feasible approximations?

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Supplementary

Link to ProvSQL: ProvSQL

Queries used: Link to queries used